Dynamical systems 2

Quantitative Methods in Historical Linguistics

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First things first: About the Portfolio

- Remember: Portfolio deadline is **1st August** (midnight)
- Please submit your Portfolio by email, as a **PDF attachment** (only one PDF please)
  - If you need to put together multiple PDFs and don't know how, just ask me!
- Your Portfolio can contain hand-written parts (you'll have to scan them)
- Remember the marking criteria. To get full points,
  1. Your solutions need to work
  2. Your solutions must not be inelegant
- Remember to include all R code (and plots) needed to answer the portfolio questions. If vital parts (parts that are necessary for replicating your results) are missing, points will be deducted
- Any questions?

From last time: A first model

- Assume a population of N speakers
- Assume that there is a binary variable of interest
  - e.g. do-support vs. no do-support
  - or final fortition vs. no final fortition
- Let's call one of these **variants** X and the other one Y
- Assume that each speaker uses either X or Y, but not both
- Let's write x_t for the number of people using X at time t and y_t for the number of people using Y at time t

From last time: A first model

- Assume that we take a look at this population once every week
  - x_0 is the number of X-speakers initially
  - x_1 is the number of X-speakers after 1 week
  - x_2 is the number of X-speakers after 2 weeks

    $\vdots$
  - x_t is the number of X-speakers after t weeks
  - And similarly for y_t
- Assume that, within one week, a proportion a of Y-speakers become X-speakers

What we found out last time

It is possible to calculate x_{t+1} and y_{t+1} if we know the values of x_t, y_t and a

Difference equations

- For example, if $x_0 = 100$, $y_0 = 200$ and $a = 0.1$, then

$$x_1 = \underbrace{100}_{=x_0} + \underbrace{0.1}_{=a} \times \underbrace{200}_{=y_0} = 100 + 20 = 120$$

  and

$$y_1 = \underbrace{200}_{=y_0} - \underbrace{0.1}_{=a} \times \underbrace{200}_{=y_0} = 200 - 20 = 180$$

- From this, we can extract the general equations:

$$\begin{cases} x_{t+1} = x_t + a y_t \\ y_{t+1} = y_t - a y_t \end{cases} \qquad (*)$$

- 'General' = valid for any time t
- (*) is a pair of **DIFFERENCE EQUATIONS** (from it you can figure out the difference between x_t and x_{t+1}, for example)

Conservation of population size

- Let's look at our pair of difference equations in a bit more detail:

$$\begin{cases} x_{t+1} = x_t + a y_t \\ y_{t+1} = y_t - a y_t \end{cases}$$

- Observation: over any time step $t \to t+1$, $a y_t$ is added to x_t and $a y_t$ is subtracted from y_t
- This means there is a **flow** from Y to X
  - The number of X-speakers should always increase
  - The number of Y-speakers should always decrease
- Moreover, population size stays constant under this flow
  - Look what happens if we calculate the total population size at time $t+1$:

$$x_{t+1} + y_{t+1} = x_t + a y_t + y_t - a y_t = x_t + y_t$$

The parameter a

- Recall that x_t and y_t are the only **VARIABLES** of this dynamical system
  - Variable of a dynamical system = something that depends on time, t
- The number a, in contrast, is a **PARAMETER**
  - It does not depend on t, but rather always stays the same

Reflection Exercise

1. Let's suppose $a = 0.5$ rather than $a = 0.1$ as previously. What happens?
2. What happens if $a = 0.01$?
3. What happens if $a = 0$?

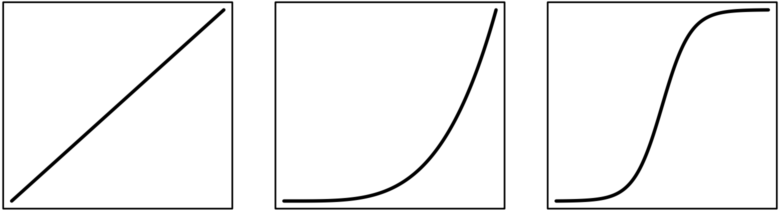
The parameter a

$$\begin{cases} x_{t+1} = x_t + a y_t \\ y_{t+1} = y_t - a y_t \end{cases}$$

- If $a = 0.5$ instead of $a = 0.1$, the number of X-speakers grows 5 times faster
- If $a = 0.01$ instead of $a = 0.1$, the number of X-speakers grows 10 times slower
- If $a = 0$, nothing happens: the number of X-speakers stays the same and the number of Y-speakers stays the same

Time evolution

- So we know the number of *X*-speakers increases over time in this model
- But **how** exactly does it increase?



Iterating a dynamical system with R

- We could calculate each x_t and y_t by hand/calculator
- But that gets very tedious... particularly if we want to do this for many **ITERATIONS** (time steps)
- Let's see how to use R for this purpose
- We need a new tool we haven't seen yet: a **LOOP**

https://commons.wikimedia.org/wiki/File:Big_Loop_Ausschnitt,_Heide-Park.jpg

Loops

- With a **FOR-LOOP**, you can make R do something more than once
- Suppose we want to print the string `"Dynamical systems are fun!"` 10 times on the screen
- Use the following loop:

```
for (i in 1:10) {
  print("Dynamical systems are fun!")
}
```

- Try it for yourself!
- What happens here? The (temporary) variable `i` goes through each value in the vector `1:10`. The `print` command is executed each time. When `i` has gone through all elements of `1:10`, R exits the loop.

Loops

- For-loops can be complicated, and they can refer to the value of `i` (which keeps changing as the loop is iterated):

```
for (i in c(1,2,3,5)) {
  j <- 2*i
  print(j)
}
```

```
## [1] 2
## [1] 4
## [1] 6
## [1] 10
```

- Can also be written in one line:

```
for (i in c(1,2,3,5)) { j <- 2*i; print(j) }
```

Iterating a dynamical system with R

- Okay, back to the topic. First, we need to decide how many time steps we want. Let's do 100:

```
iterations <- 100
```

- Then we need vectors to hold values of x_t and y_t:

```
x <- rep(NA, length=iterations)
y <- rep(NA, length=iterations)
```

- Put the initial values in place. For example, let's do $x_0 = 10$ and $y_0 = 590$:

```
x[1] <- 10
y[1] <- 590
```

- We also need a value of a. Let's stick to $a = 0.1$:

```
a <- 0.1
```

Iterating a dynamical system with R

- Now the `for` loop:

```
for (t in 1:(iterations-1)) {
  x[t+1] <- x[t] + a*y[t]
  y[t+1] <- y[t] - a*y[t]
}
```

- Note how this mirrors our difference equations:

$$\begin{cases} x_{t+1} = x_t + ay_t \\ y_{t+1} = y_t - ay_t \end{cases}$$

- The magic here is: `x` and `y` are both vectors, and `t` is used to index them. The $(t+1)$th value of the `x` vector is calculated from the tth value of `x` and `y`, and similarly for `y`.

Iterating a dynamical system with R

- It's best to put this in a function

```
first_model <- function(iterations, x0, y0, a) {
  x <- rep(NA, length=iterations)
  y <- rep(NA, length=iterations)
  x[1] <- x0
  y[1] <- y0
  for (t in 1:(iterations-1)) {
    x[t+1] <- x[t] + a*y[t]
    y[t+1] <- y[t] - a*y[t]
  }
  times <- 0:(iterations-1)
  data.frame(t=times, x=x, y=y)
}
```

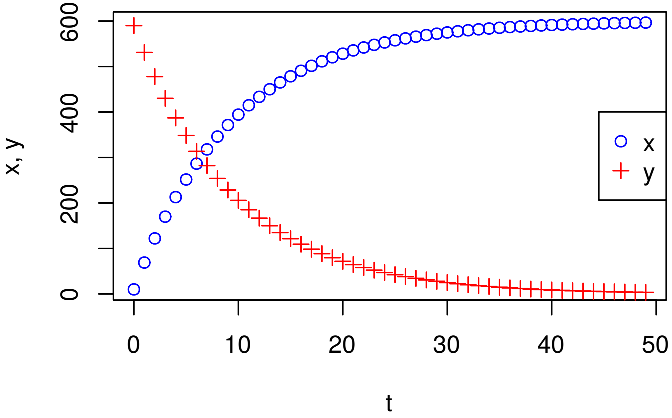
Iterating a dynamical system with R

- Now we can use this function:

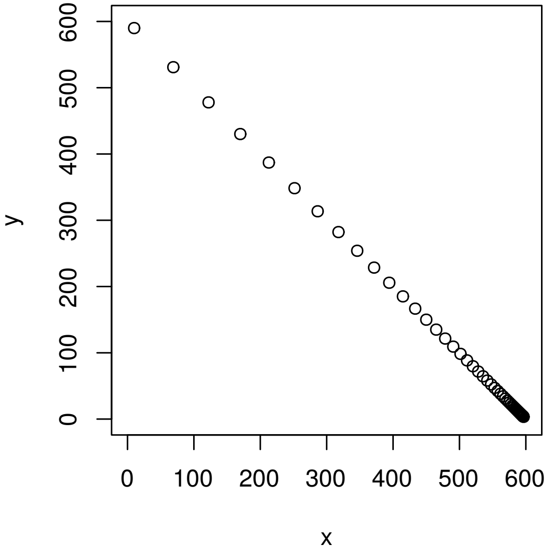
```
df <- first_model(iterations=50, x0=10, y0=590, a=0.1)
head(df)

##   t        x        y
## 1 0  10.0000 590.0000
## 2 1  69.0000 531.0000
## 3 2 122.1000 477.9000
## 4 3 169.8900 430.1100
## 5 4 212.9010 387.0990
## 6 5 251.6109 348.3891
```

Iterating a dynamical system with R

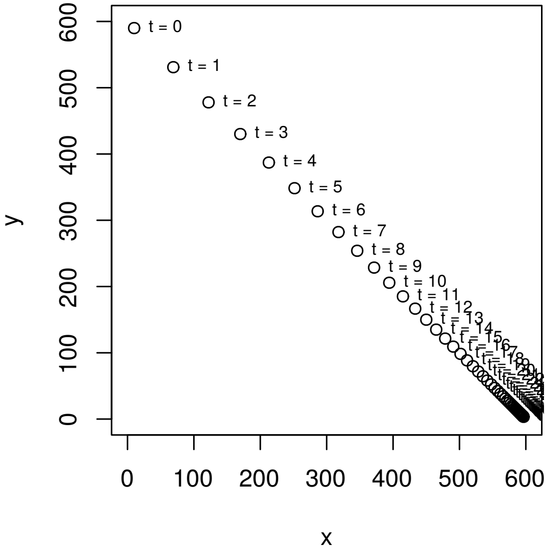
```
plot(df$t, df$x, xlab="t", ylab="x, y", col="blue")
points(df$t, df$y, pch=3, col="red")
legend("right", pch=c(1,3), col=c("blue", "red"),
       legend=c("x", "y"))
```



The state space



The state space



The state space

- This kind of plot is known as a **STATE SPACE PLOT**
- Making one is not difficult:

```
plot(df$x, df$y, xlim=c(0,600), ylim=c(0,600),
     xlab="x", ylab="y")
```

- If you want the time labels, add the following command:

```
text(x=df$x, y=df$y, labels=paste("t =", df$t),
     pos=4, cex=0.7)
```

  - `text` puts `labels` at the specified `x` and `y` positions
  - `pos=4` means labels should go to the right of the points
  - `cex=0.7` (“character expansion”) makes the text a bit smaller
  - As always, use `?text` (or Google) to find out more

A fixed point

- The state to which this system tends as t grows is given by the solution of the following equations:

$$\begin{cases} x_t = x_t + a y_t \\ y_t = y_t - a y_t \end{cases}$$

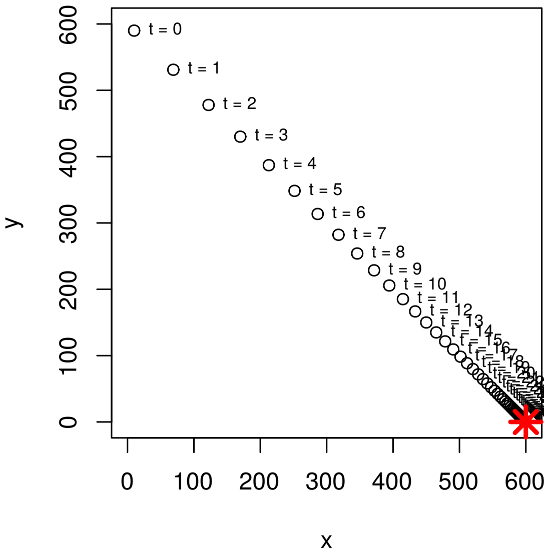
(i.e. we specify $x_{t+1} = x_t$ and $y_{t+1} = y_t$)

- The only solution is $x = N$, $y = 0$:

$$\begin{cases} N = N + a \times 0 = N \\ 0 = 0 - a \times 0 = 0 \end{cases}$$

- The state $x = N$, $y = 0$ is known as a **FIXED POINT** (or a **REST POINT**)
- This system has only one fixed point, but in general dynamical systems may have several fixed points

A fixed point



A second model: Two flows

- In reality, we'd expect some X-speakers to become Y-speakers, too, and not just vice versa
- I.e. we have two flows

Reflection Exercise

Suppose

- Y-speakers become X-speakers at a rate a (as before)
- X-speakers become Y-speakers at a rate b

What is the pair of difference equations for this extended model?

A second model: Two flows

- The equations are:

$$\begin{cases} x_{t+1} = x_t + ay_t - bx_t \\ y_{t+1} = y_t - ay_t + bx_t \end{cases}$$

- Rearranging the terms a bit, we have:

$$\begin{cases} x_{t+1} & = & (1-b)x_t & + & ay_t \\ y_{t+1} & = & bx_t & + & (1-a)y_t \end{cases}$$

- The dynamics are now determined by a and b
- Questions we must answer:
  1. Which one increases, x_t or y_t?
  2. What are the eventual values of x_t and y_t (when t is large)?
  3. What **SHAPE** do the dynamics take in the time dimension?

Fixed point(s)

- So now we have the slightly more complicated pair of equations:

$$\begin{cases} x_{t+1} = (1-b)x_t + ay_t \\ y_{t+1} = bx_t + (1-a)y_t \end{cases}$$

- How to, first of all, figure out the fixed point(s) of this system?
- Insight: since population size is conserved, we always have $y_t = N - x_t$
- So we only need one equation, really:

$$x_{t+1} = (1-b)x_t + a(\underbrace{N - x_t}_{=y_t})$$

Fixed point(s)

- Need to solve this equation for x to get the fixed point(s):

$$x = (1 - b)x + a(N - x)$$

Fixed point(s)

- Need to solve this equation for x to get the fixed point(s):

$$x = (1 - b)x + a(N - x)$$

- Let's go step by step:

$$x = x - bx + aN - ax$$

Fixed point(s)

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$$x - aN = x - bx - ax$$

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$$x - aN = x - bx - ax$$

$$-aN = -bx - ax$$

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$$aN = bx + ax$$

Fixed point(s)

- Need to solve this equation for x to get the fixed point(s):

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- Let's go step by step:

$$x = x - bx + aN - ax$$

$$x - aN = x - bx - ax$$

$$-aN = -bx - ax$$

$$aN = bx + ax$$

$$aN = (a + b)x$$

Fixed point(s)

- Need to solve this equation for x to get the fixed point(s):

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- Let's go step by step:

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$$x - aN = x - bx - ax$$

$$-aN = -bx - ax$$

$$aN = bx + ax$$

$$aN = (a + b)x$$

$$(a + b)x = aN$$

Fixed point(s)

- Need to solve this equation for x to get the fixed point(s):

$$x = (1 - b)x + a(N - x)$$

- Let's go step by step:

$$x = x - bx + aN - ax$$

$$x - aN = x - bx - ax$$

$$-aN = -bx - ax$$

$$aN = bx + ax$$

$$aN = (a + b)x$$

$$(a + b)x = aN$$

$$x = \frac{aN}{a + b}$$

Fixed point(s)

- So again there is just one fixed point:

$$x = \frac{aN}{a+b}$$

- But this time its value depends on a and b

Reflection Exercise

1. What happens if $b = 0$ (but $a \neq 0$)?
2. What happens if $a = 0$ (but $b \neq 0$)?
3. What happens if $a = b$ (but $a, b \neq 0$)?

Fixed point(s)

1. If $b = 0$ (but $a \neq 0$):

$$x = \frac{aN}{a+b} = \frac{aN}{a} = N$$

We get the same fixed point we saw with the first model (makes sense!)

2. If $a = 0$ (but $b \neq 0$):

$$x = \frac{aN}{a+b} = \frac{0}{b} = 0$$

This is different. Also makes sense.

3. If $a = b$ (but $a, b \neq 0$):

$$x = \frac{aN}{a+b} = \frac{aN}{a+a} = \frac{aN}{2a} = \frac{N}{2}$$

If both transition rates $X \rightarrow Y$ and $Y \rightarrow X$ are equal, the fixed point is at half the population size. Makes sense!

Bye

- Next week (18th July) is our last session
- (But there will still be a workshop on the 25th)
- Remember: Portfolio deadline is **1st August**
- There's a new Portfolio Exercise on ILIAS
- And there will be one more next week, and that's it
- Any questions?